Normalization Factors, Reflection Amplitudes and Integrable Systems

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Abstract

We calculate normalization factors and reflection amplitudes in the W-invariant conformal quantum field theories. Using these CFT data we derive vacuum expectation values of exponential fields in affine Toda theories and related perturbed conformal field theories. We apply these results to evaluate explicitly the expectation values of order parameters in the field theories associated with statistical systems, like XY, Z_n-Ising and Ashkin-Teller models. The same results are used for the calculation of the asymptotics of cylindrically symmetric solutions of the classical Toda equations which appear in topological field theories. The integrable boundary Toda theories are considered. We derive boundary reflection amplitudes in non-affine case and boundary one point functions in affine Toda theories. The boundary ground state energies are cojectured. In the last section we describe the duality properties and calculate the reflection amplitudes in integrable deformed Toda theories.

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1 Introduction

There is a large class of 2D quantum field theories (QFTs) which can be considered as perturbed conformal field theories (CFTs). These theories are completely defined if one specifies its CFT data and the relevant operator which plays the role of perturbation. The CFT data contain explicit information about the ultraviolet (UV) asymptotics of the field theory while the long distance behavior is the subject of analysis. If a perturbed CFT contains only massive particles it is equivalent to the relativistic scattering theory and is completely defined specifying the S-matrix. Contrary to CFT data the S-matrix data exhibit the information about the long distance property of the theory in the explicit way while the UV asymptotics have to be derived.

A link between these two kinds of data would provide a good view point for the understanding of the general structure of 2D QFT. In general this problem does not look tractable. Whereas the CFT data can be specified in a relatively simple way the general S-matrix is rather complicated object even in two dimensions. However, there exists rather important class of 2D QFTs (integrable theories) where the scattering theory is factorized and S-matrix can be described in great details. These theories admit rather complete description in the UV and IR regimes.

In these proceedings we consider the application of special CFT data (normalization factors and reflection amplitudes) to the analysis of integrable QFTs. The normalization factors appear in the CFTs which possess the representation for the primary fields in terms of vertex operators and are used for the proper normalization of the fields. We show that rather important information about the integrable perturbed CFTs is encoded at these data. As an example of integrable QFT we study the simply-laced affine Toda theory (ATT), which can be considered as the perturbed CFT (non-affine Toda theory). This CFT possesses the extended symmetry generated by the W-algebra. We calculate the normalization factors (NFs) and derive the reflection amplitudes for W-invariant CFTs. We use the normalization factors to relate the coupling constants and the masses of particles in perurbed CFTs.

The reflection amplitudes (RAs) in CFT define the linear transformations between the different exponential fields corresponding to the same primary field of the full symmetry algebra of theory. They play the crucial role in the calculation of the vacuum expectation values of the exponential fields in perturbed CFTs [2] as well as for the analysis of UV asymptotics of the observables in these QFTs [3],[4]. In section 3 we use these reflection amplitudes to calculate the one point functions in ATT and related perturbed CFTs. These results are applied to the explicit evaluation of the vacuum expectation values of the order parameters (spins) in critical Z_n Ising and Ashkin-Teller models perturbed by the thermal operators. In section 4 we use the vacuum expectation values of the exponential fields in ATT to derive the asymptotics of the cylindrically symmetric solutions of the classical Toda equations which appear in the analysis of topological QFTs [6].

In section 5 we consider integrable boundary Toda theories. We derive the boundary reflection amplitudes in the boundary non-affine Toda theories (NATTs) and vacuum expectation values of boundary exponential fields in ATTs. Using these expectation values we calculate the classical boundary ground state energy and give the conjecture for it in the quantum case. The bounday scattering theory which is consistent with this conjecture is discussed. In the last section we describe the duality properties and derive the reflection amplitudes for integrable deformed Toda theories, which have some physical applications.

2 Affine and Non-Affine Toda Theories, Normalization Factors and Reflection Amplitudes

The ATT corresponding to the Lie algebra G of rank r is described by the action:

$$\mathcal{A}_b = \int d^2x \left[\frac{1}{8\pi}(\partial_\mu \varphi)^2 + \mu \sum_{i=1}^{r} e^{be_i\cdot\varphi} + \mu e^{be_0\cdot\varphi} \right], \tag{1}$$

where e_i, $i = 1, ..., r$ are the simple roots of Lie algebra G and $-e_0$ is a maximal root:

$$e_0 + \sum_{i=1}^{r} n_i e_i = 0. \tag{2}$$

The fields φ in eq. (1) are normalized so that at $\mu = 0$

$$\langle \varphi_a(x), \varphi_b(y) \rangle = -\delta_{ab} \log |x - y|^2 \tag{3}$$

We will consider later the simply-laced Lie algebras A, D, E.

For real b the spectrum of these ATTs consists from r particles with masses m_i ($i = 1, ..., r$) given by:

$$m_i = m\nu_i, \qquad m^2 = \frac{1}{2h} \sum_{i=1}^{r} m_i^2, \tag{4}$$

where h is Coxeter number of G and ν_i^2 are the eigenvalues of the mass matrix:

$$M_{ab} = \sum_{i=1}^{r} n_i (e_i)^a (e_i)^b + (e_0)^a (e_0)^b. \tag{5}$$

The exact relation between the parameter m characterizing the spectrum of physical particles and the parameter μ in the action (1) can be obtained by the Bethe ansatz method (see for example [7],[8]). It can be easily derived from the results of paper [8] and has the form:

$$-\pi\mu\gamma(1+b^2) = \left[\frac{mk(G)\Gamma\left(\frac{1}{(1+b^2)h}\right)\Gamma\left(\frac{b^2}{(1+b^2)h}\right)b^2}{2\Gamma\left(\frac{1}{h}\right)h(1+b^2)}\right]^{2(1+b^2)}, \tag{6}$$

where as usual $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ and

$$k(G) = \left(\prod_{i=1}^{r} n_i^{n_i}\right)^{1/2h} \tag{7}$$

with n_i defined by the equation (2).

The ATTs can be considered as the perturbed CFTs. Without the last term (with root e_0) the action (1) describes NATTs, which are conformal. To describe the generator of the conformal symmetry we introduce the complex coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ and vector

$$Q = (b + 1/b)\rho, \qquad \rho = \frac{1}{2}\sum_{\alpha>0} \alpha, \tag{8}$$

where the sum in the definition of the Weyl vector ρ runs over all positive roots α of Lie algebra G.

The holomorphic stress energy tensor

$$T(z) = -\frac{1}{2}(\partial_z \varphi)^2 + Q \cdot \partial_z^2 \varphi \tag{9}$$

ensures the local conformal invariance of the NATT with the central charge

$$c = r + 12Q^2 = r(1 + h(h+1)(b+1/b)^2). \tag{10}$$

Besides the conformal invariance NATT possesses the additional symmetry generated by two copies of the chiral $W(G)$-algebras: $W(G) \otimes \overline{W}(G)$. The full chiral $W(G)$-algebra contains r holomorphic fields $W_j(z)$ ($W_2(z) = T(z)$) with spins j which follow the exponents of Lie algebra G. The explicit representation of these fields in terms of fields $\partial_z \varphi_a$ can be found in [9]. The primary fields Φ_w of $W(G)$-algebra are specified by r eigenvalues w_j, $j = 1, ..., r$ of the operators $W_{j,0}$ (the zeroth Fourier components of the currents $W_j(z)$):

$$W_{j,0}\Phi_w = w_j\Phi_w; \qquad W_{j,n}\Phi_w = 0, \quad n > 0. \tag{11}$$

The exponential fields:

$$V_a(x) = e^{a\cdot\varphi(x)} \tag{12}$$

are spinless conformal primary fields with the dimensions:

$$\Delta(a) = w_2(a) = \frac{Q^2}{2} - \frac{(a-Q)^2}{2}. \tag{13}$$

The fields $V_a(x)$ are also primary fields with respect to all chiral algebra $W(G)$ with the eigenvalues w_j depending on a. The functions $w_j(a)$ possess the symmetry with respect to Weyl group $\mathcal{W}$ of Lie algebra G [9] which acts at the vector a as:

$$a \to s(a) = Q + \hat{\mathbf{s}}(a-Q)\ , \quad \hat{\mathbf{s}} \in \mathcal{W}; \qquad w_j(s(a)) = w_j(a). \tag{14}$$

It means that the fields $V_{s(a)}$ for different $s \in \mathcal{W}$ are the reflection image of each other and are related by the linear transformation:

$$V_a(x) = R_s(a)V_{s(a)}(x), \tag{15}$$

where $R_s(a)$ is the "reflection amplitude".

To calculate the function $R_s(a)$ we introduce the normalized primary fields Φ_w :

$$\Phi_w = N^{-1}(a)V_a(x), \tag{16}$$

where the normalization factor $N(a)$ is chosen in the way that field Φ_w satisfies conformal normalization condition:

$$\langle\Phi_w(x), \Phi_w(y)\rangle = \frac{1}{|x-y|^{4\Delta}}. \tag{17}$$

The normalized fields Φ_w are invariant under the reflection transformations and hence:

$$R_s(a) = \frac{N(a)}{N(s(a))}. \tag{18}$$

For the calculation of the normalization factor $N(a)$ we can use the integral representation for the correlation functions of the $W(G)$ -invariant CFT (see [9] for details). We note that the operators $\hat{Q}_i$ defined as:

$$\hat{Q}_i = \int d^2x e^{be_i\cdot\varphi(x)} \tag{19}$$

commute with all of the elements of $W(G)$-algebra and can be used for the calculation of the correlation functions in NATT. In particular, if vector a satisfies the condition

$$2a+\sum_{i=1}^{r}k_ie_i=0 \tag{20}$$

with non-negative integer k_i we obtain from eqs.(16, 17) the following expression for the function $N(a)$ in terms of Coulomb integrals [9]:

$$N^2(a)=|x|^{4\Delta}\left\langle V_a(x)V_a(0)\prod_{i=1}^{r}\frac{\hat{Q}_i^{k_i}}{k_i!}\right\rangle \tag{21}$$

where the expectation value in eq.(21) is taken over the Fock vacuum of massless fields φ with the correlation functions (3).

The normalization integral (21) can be calculated and the result has the form:

$$\begin{aligned} N^2(a) &= (\pi\mu\gamma(b^2))^{-2\rho\cdot a/b} \\ &\times\prod_{\alpha>0}\frac{\Gamma(1+Q_\alpha/b)\Gamma(1+Q_\alpha b)\Gamma(1+\bar{a}_\alpha/b)\Gamma(1+\bar{a}_\alpha b)}{\Gamma(1-Q_\alpha/b)\Gamma(1-Q_\alpha b)\Gamma(1-\bar{a}_\alpha/b)\Gamma(1-\bar{a}_\alpha b)} \end{aligned} \tag{22}$$

where:

$$\bar{a}=(a-Q),\quad Q_\alpha=Q\cdot\alpha,\quad \bar{a}_\alpha=(a-Q)\cdot\alpha \tag{23}$$

and the product runs over all positive roots α of Lie algebra G.

With these normalization factors the integral representation with screening charges [9] reproduces correlation functions of the primary fields Φ_w (16) satisfying the conformal normalization condition (17). We accept eq.(22) as the proper analytical continuation of the function $N^2(a)$ for all a. It gives the following expression for the reflection amplitude $R_s(a)$:

$$R_s(a)=\frac{N(a)}{N(s(a))}=\frac{A_{s(a)}}{A_a}, \tag{24}$$

where

$$A(a)=(\pi\mu\gamma(b^2))^{\rho\cdot\bar{a}/b}\prod_{\alpha>0}\Gamma(1-\bar{a}_\alpha/b)\Gamma(1-\bar{a}_\alpha b). \tag{25}$$

With this function the reflection relation (15) can be written in more symmetric form as:

$$A_aV_a(x)=A_{s(a)}V_{s(a)}(x),\qquad s(a)=Q+\hat{\mathbf{s}}(a-Q),\quad \hat{\mathbf{s}}\in\mathcal{W}. \tag{26}$$

The action $\mathcal{A}_\beta$ of the imaginary ATT can be obtained from the action (1) by the substitution $b \to i\beta, \quad \mu \to -\mu$. This action is invariant under the transformation $\varphi \to \varphi + 2\pi\theta/\beta$, where θ belongs to the weight lattice of G. It means that the space of the vacua of imaginary ATT is equivalent to this lattice. In the classical case this theory possesses the solitons. The basic solitons have the topological charge proportional to the weights of the fundamental representations $\pi_i(G)$. In the quantum case the particles, corresponding to these solitons form the multiplets which can be associated with the fundamental representations of the Yangian $Y_i(G)$. The masses M_i $(i = 1, ..., r)$ of the particles in these multiplets have the form: $M_i = M\nu_i$, where mass parameter M and eigenvalues ν_i are defined by the eqs.(4,5). The exact relation between the parameters of the action $\mathcal{A}_\beta$ and the physical mass M can be obtained by the Bethe ansatz method [7],[8] and has the form:

$$\pi\mu\gamma(1-\beta^2) = \left[\frac{M\pi k(G)\Gamma\left(\frac{1}{(1-\beta^2)h}\right)}{\Gamma\left(\frac{1}{h}\right)\Gamma\left(\frac{\beta^2}{(1-\beta^2)h}\right)}\right]^{2(1-\beta^2)}. \tag{27}$$

The imaginary ATTs are non-unitarian QFTs. However, for special vealues of $\beta^2 = p/(p+1)$ with integer $p > h$, these QFTs admit the restriction with respect to the affine quantum group $U_q(\hat{G})$ with level equal to zero and $q = \exp(i\pi/\beta^2)$ to the unitarian QFTs [10],[11]. This restricted theories can be considered as the minimal unitarian models $\mathcal{M}_p(G)$ of the $W(G)$- invariant CFTs perturbed by the relevant operator $\Phi_{ad} \in \mathcal{M}_p(G)$ associated with the adjoint representation of G.

The minimal models $\mathcal{M}_p(G)$ are described in details in [9]. They are characterized by the central charges:

$$c_p(G) = r\left(1 - \frac{h(h+1)}{p(p+1)}\right), \quad p > h. \tag{28}$$

The primary fields $\Phi(\Omega, \Omega') \in \mathcal{M}_p(G)$ are specified by two highest weight vectors Ω and Ω' which satisfy the conditions: $-\Omega \cdot e_0 \leq p + 1 - h$ and $-\Omega' \cdot e_0 \leq p - h$. The fields $\Phi_\Omega = \Phi(\Omega, 0)$ and $\Phi_{\Omega'} = \Phi(0, \Omega')$ form two closed operator subalgebras of $\mathcal{M}_p(G)$ CFTs. The primary, strongly degenerate fields $\Phi(\Omega, \Omega')$, normalized by the condition (17) can be represented in terms of the fields φ of imaginary NATT by the relation:

$$\Phi(\Omega, \Omega') = \mathcal{N}^{-1}(a)e^{ia\cdot\varphi}, \quad a = -\beta\Omega + \Omega'/\beta, \quad \beta^2 = p/(p+1). \tag{29}$$

The normalization factors $\mathcal{N}(a)$ can be obtained from the NFs $N(a)$ (22) by the substitutions:

$$b \to i\beta, \quad a \to ia, \quad \mu \to -\mu, \quad Q \to iQ_\beta = i(\beta - 1/\beta)\rho. \tag{30}$$

Being rather complicated for general values of a, the NFs $\mathcal{N}(a)$ and $N(a)$ simplify for some particular values of a, which are useful for practical applications. For example, it happens if $a = -\beta\omega_i$, where ω_i are the fundamental weights of G ($\omega_i \cdot e_j = \delta_{ij}$) satisfying the condition: $-e_0 \cdot \omega_i = 1$ (or n_i in the eq.(2) is equal to 1). In this case the normalization factors have the form which is very similar to Weyl formula (modulo γ-functions) for the inverse dimensions of representations $\pi_i(G)$:

$$\mathcal{N}^2(-\beta\omega_i) = (\pi\mu\gamma(u))^{2\omega_i\rho} \prod_{\alpha>0} \frac{\gamma(\rho_\alpha u)}{\gamma((\rho+\omega_i)_\alpha u)}; \quad u = 1-\beta^2. \tag{31}$$

In particular, the NFs $\mathcal{N}(-\beta\omega_i)$ for Lie algebra A_{n-1} (where all fundamental weights satisfy this condition) can be written as:

$$\mathcal{N}^2(-\beta\omega_k) = (\pi\mu\gamma(u))^{k(n-k)} \prod_{j=1}^{k} \frac{\gamma(ju)}{\gamma((n+1-j)u)}, \tag{32}$$

For the perturbing operator $\Phi_{ad} \equiv \Phi_{\Omega_{ad}} = \Phi_{-e_0}$ the normalization factor for all simply-laced cases can be written as:

$$\mathcal{N}^2(\beta e_0) = (\pi\mu\gamma(u))^{2h-2} \frac{(1-(h+1)u)^2\gamma(u)\gamma(qu)\gamma(\frac{h-2q+2}{2}u)}{(1-hu)^2\gamma(\frac{h+2q-2}{2}u)\gamma((h-q)u)\gamma((h+1)u)}$$

here $u = 1-\beta^2 = 1/(p+1)$ and $q = \max_i n_i$, where n_i are defined by the eq.(2): $q(A) = 1,\ q(D) = 2,\ q(E_6) = 3,\ q(E_7) = 4,\ q(E_8) = 6$.

The action of the perturbed CFT $\mathcal{M}_p(G)$ can be written in the form:

$$\mathcal{A}_p = \mathcal{A}_{CFT} + \lambda \int d^2x \Phi_{ad}(x). \tag{33}$$

The perturbing operator normalized by eq.(17) can be represented using eq.(29) as: $\Phi_{ad} = \mathcal{N}^{-1}(\beta e_0)\exp(i\beta e_0 \cdot \varphi)$. This exponential field is invariant under the quantum group restriction of ATT [10]. Comparing the action (33) with the action of the imaginary ATT we can write $\lambda = -\mu\mathcal{N}(\beta e_0)$. Expressing μ in terms of the physical mass parameter M by eq.(27) we obtain the exact relation between coupling constant λ and M [8]:

$$\begin{aligned} \lambda^2 &= \left[\frac{M\pi k(G)\Gamma\left(\frac{p+1}{h}\right)}{\Gamma\left(\frac{1}{h}\right)\Gamma\left(\frac{p}{h}\right)}\right]^{4hu} \\ &\times \frac{(1-hu)^{-2}(1-(h+1)u)^2\gamma(qu)\gamma(\frac{h-2q+2}{2}u)}{\pi^2\gamma(u)\gamma(\frac{h+2q-2}{2}u)\gamma((h-q)u)\gamma((h+1)u)}. \end{aligned} \tag{34}$$

This relation permits us to express the UV perturbative expansions in coupling constant λ in terms of the IR parameter M.

3 Expectation Values of Local Fields in ATT and Related Perturbed CFT

The vacuum expectation values (VEVs) of local fields play an important role in the QFT and statistical mechanics. In statistical mechanics the VEVs determine the "generalized susceptibilities", i.e. the linear response of the system to external fields. In the QFT defined as a perturbed CFT the VEVs provide all the information about its correlation functions that is not accessible through direct calculation in conformal perturbation theory [13]. Recently, some progress was made in the calculation of the VEVs in two dimensional integrable QFTs [1],[2]. Namely, it was shown in [2] that VEVs of the exponential fields $V_a(x)$ in perturbed CFTs satisfy the same "reflection relations" as the vertex operators $V_a(x)$ in basic CFT.

We define the function $G(a)$ as the vacuum expectation value of the operator $\exp(a\cdot\varphi)$ in ATTs with real coupling b.

$$G(a) = \langle \exp(a\cdot\varphi)\rangle_b \, . \tag{35}$$

For any element $\hat{\mathbf{s}}$ of Weyl group $\mathcal{W}$ this function satisfies the functional equation:

$$A_a G(a) = A_{s(a)} G(s(a)), \quad s(a) = Q + \hat{\mathbf{s}}(a-Q), \quad \hat{\mathbf{s}} \in \mathcal{W} \tag{36}$$

where function A_a is given by eq.(25).

The minimal meromorphic solution to these functional equations which respects all the symmetries of extended Dynkin diagrams of Lie algebras ADE has the form [12]:

$$\begin{aligned} G(a) \;\; &= \;\; \left[\frac{mk(G)\Gamma\left(\frac{1}{(1+b^2)h}\right)\Gamma\left(\frac{b^2}{(1+b^2)h}\right)b^2}{2\Gamma\left(\frac{1}{h}\right)h(1+b^2)}\right]^{-a^2} \\ &\quad \times \exp\left(\int \frac{dt}{t}[a^2e^{-2t} - F_b(a,t)]\right) \end{aligned} \tag{37}$$

where:

$$F_b(a,t) \;\; = \;\; \sinh((1+b^2)t)$$

$$\times \sum_{\alpha>0} \frac{\sinh(ba_\alpha t)\sinh((b(a-2Q)_\alpha + h(1+b^2))t)}{\sinh t \sinh(b^2 t)\sinh((1+b^2)ht)}. \qquad (38)$$

This solution satisfies many possible perturbative and non-perturbative tests for one point function in ATTs, which we will not discuss here (some non-perturbative tests will be considered bellow).

The VEVs of the exponential fields in imaginary ATTs:

$$\mathcal{G}(a) = \langle \exp(ia\cdot\varphi)\rangle_\beta \qquad (39)$$

can be obtained from function $G(a)$ by the the substitution (30). Being expressed in terms of physical mass parameter M, it has a form:

$$\mathcal{G}(a) = \left[\frac{M\pi k(G)\Gamma\left(\frac{1}{(1-\beta^2)h}\right)}{\Gamma\left(\frac{1}{h}\right)\Gamma\left(\frac{\beta^2}{(1-\beta^2)h}\right)}\right]^{a^2} \exp\left(-\int \frac{dt}{t}[a^2 e^{-2t} - \mathcal{F}_\beta(a,t)]\right) \qquad (40)$$

with

$$\mathcal{F}_\beta(a,t) = \sinh u \sum_{\alpha>0} \frac{\sinh(\beta a_\alpha t)\sinh((\beta(a-2Q_\beta)_\alpha - hu)t)}{\sinh t \sinh(\beta^2 t)\sinh(uht)} \qquad (41)$$

where $u = 1-\beta^2$ and $Q_\beta = (\beta - 1/\beta)\rho$.

Being rather complicated for general a functions $\mathcal{F}_\beta(a,t)$ and $F_b(a,t)$ simplify drastically for special directions of vector a, which are useful for the practical applications. Namely, for $a = v\omega_k$, where ω_k is a fundamental weight of G, satisfying the condition: $-e_0\cdot\omega_k = 1$, function $\mathcal{F}_\beta(a,t)$ can be written as:

$$\mathcal{F}_\beta(v\omega_k,t) = \frac{\sinh^2(\beta v t)}{\sinh t\sinh(\beta^2 t)}\left(2\cosh(tu)\delta_{ij} - \mathbf{I}_{ij}\right)^{-1}_{kk}. \qquad (42)$$

where the matrix $\mathbf{I}_{ij} = 2\delta_{ij} - e_i\cdot e_j$. In particular, for Lie algebra A_{n-1} all fundamental weights satisfy this condition and we have:

$$\mathcal{F}_\beta(v\omega_k,t) = \frac{\sinh^2(\beta v t)\sinh(kut)\sinh((n-k)ut)}{\sinh t\sinh(\beta^2 t)\sinh(ut)\sinh(nut)}. \qquad (43)$$

As an example of the application of eq.(40), we consider the particular correlations in A_{n-1} imaginary ATT in the limit $n\to\infty$. In this limit this QFT can be interpreted as the special case of 3D $U(1)$ or XY model. The action of this QFT can be written in terms of the fields $U_k(x) = \exp(i\beta\varphi_k(x)) \in U(1)$, $k = 1, ...n$, which satisfy periodic boundary condition: $U_1(x) = U_n(x)$. Namely:

$$\mathcal{A}_\beta = \int d^2x \left(\sum_{k=1}^{n} \frac{1}{8\pi\beta^2} \partial_\mu U_k \partial_\mu U_k^{-1} + \mu U_k U_{k+1}^{-1} \right). \tag{44}$$

The model is continuous in two dimensions and discrete in the third. Using eq.(40) we can calculate the correlations between the fields $U_k(x)$ taken at the same point x and at different k. At the limit $n \to \infty$ we obtain that for $l > 0$:

$$\left\langle U_{k\pm l}(x) U_k^{-1}(x) \right\rangle = \left(\frac{M\beta^2}{2} \right)^{2\beta^2} \Gamma^2(u) \frac{\Gamma(2+(l-1)u)\Gamma(1+(l-1)u)}{\Gamma(1+lu)\Gamma(2+(l-2)u)}. \tag{45}$$

The vacuum expectation values (40) can be used for the calculations of the VEVs of the primary operators in perturbed CFTs $\mathcal{M}_p(G)$ (33). Here we consider only the case of the subalgebra Φ_Ω. The primary fields from this subalgebra are represented by the exponential fields (29) (with Ω'=0) which are invariant under the quantum group restriction [2] and their VEVs can be easily expressed through the function $\mathcal{G}(a)$ and NF $\mathcal{N}(a)$. The general case we suppose to consider in the separate publication.

The QFT (33) contains the finite set $\{s\}$ of the degenerate vacuum states, which can be specified by the highest weights θ_s satisfying the condition:

$$-e_0 \cdot \theta_s \leq p - h. \tag{46}$$

The particles in this QFT are the kinks interpolating between different vacua. The masses of these excitations coincide with the masses of basic particles in imaginary ATT and are related with coupling constant λ by eq.(34). The vector θ_s determines the shift of the field $\varphi: \ \varphi \to \varphi + 2\pi\theta_s/\beta$. corresponding to vacuum state s. As a result we obtain from the eq.(29) the following expression for the VEVs of the normalized fields Φ_Ω in the QFT (33):

$$\langle \Phi_\Omega \rangle_s = \exp(-i2\pi\theta_s \cdot \Omega) \mathcal{N}^{-1}(-\beta\Omega) \mathcal{G}(-\beta\Omega). \tag{47}$$

As an example, we apply this equation to calculate the vacuum expectation values of the spin field (order parameter) σ in the critical Z_n-Ising models perturbed by the first thermal operator ε. Z_n-Ising models are the natural generalizations of Ising model to the case when spin variable takes its values in group Z_n (see ref. [14] for details). They are self-dual, i.e. possess Kramers-Wannier symmetry. At the self-dual manifold of parameters Z_n-Ising models have the critical points, which were found in [14]. At these critical points Z_n-Ising models are described by the Z_n-parafermionic CFTs with central charge $c = \frac{2(n-1)}{n+2}$. Besides the parafermionic symmetry these

CFTs possess also $W(A_{n-1})$ symmetry and can be described by $\mathcal{M}_{n+1}(A_{n-1})$ minimal models with $\beta^2 = \frac{n+1}{n+2}$ $(u = \frac{1}{n+2})$. The spin field σ in critical Z_n-Ising models has the dimension $\Delta = \frac{(n-1)}{2n(n+2)}$ and coincides with the primary field $\Phi_{\omega_1} \in \mathcal{M}_{n+1}(A_{n-1})$. The perturbing operator Φ_{ad} is exactly the first thermal operator ε with dimension $\delta = \frac{2}{(n+2)}$. The operator ε is anti self-dual, i.e. it changes the sign under duality transformation. It means that depending on the sign of λ in eq.(33) the perturbed theory will be in ordered or disordered phase. In the first (second) case the order (disorder) parameter σ (μ) has non-zero VEVs.

In the ordered phase the vacua $\{s\}$ are specified by the highest weights θ_s which satisfy the inequality (46), which can be written as: $-e_0 \cdot \theta_s \leq 1$. This condition has n solutions $\theta_s = \{0, \omega_1, ..., \omega_{n-1}\}$, so, we have for the first factor in eq.(47): $\exp(-2\pi i\theta_s \cdot \omega_1) = \exp(i2\pi s/n)$, $s = 0, ...n-1$. Taking $\mathcal{N}^{-1}(-\beta\omega_1)$ and $\mathcal{G}(-\beta\omega_1)$ from eqs.(32,43) with $u = 1 - \beta^2 = \frac{1}{n+2}$ we obtain the final expression for VEV of the order parameter σ:

$$\begin{aligned} \langle\sigma\rangle_s &= \exp(i2\pi s/n) \left[\frac{M\pi\Gamma\left(\frac{n+2}{n}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n+1}{n}\right)} \right]^{2\Delta} \left(\frac{\gamma(\frac{n}{n+2})}{\gamma(\frac{1}{n+2})} \right)^{\frac{1}{2}} \\ &\times \exp\left(\int \frac{dt}{t} \left(\frac{\sinh(\beta^2 t)\sinh((n-1)ut)}{\sinh t \sinh(nut)} - \frac{n-1}{n}\beta^2 e^{-2t} \right) \right) \end{aligned} \quad (48)$$

For odd $n = 2l + 1$ the integral in eq (48) can be calculated and result has a form:

$$\langle\sigma\rangle_s = e^{i2\pi s/n} \left[\frac{M\pi\Gamma\left(\frac{n+2}{n}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n+1}{n}\right)} \right]^{2\Delta} \left(\frac{n+2}{n} \right)^{\frac{n-1}{2n}} \left(\frac{\gamma(\frac{n}{n+2})}{\gamma(\frac{1}{n+2})} \right)^{\frac{1}{2}} \prod_{j=1}^{l} \frac{\gamma(\frac{2j-1}{n+2})}{\gamma(\frac{2j+1}{n+2})}.$$

We note that particles in this theory can be considered as the solitons connecting different vacua. In particular, their masses are proportional to distances between these vacua: $M_k \sim |\exp(2\pi i(s+k)/n) - \exp(2\pi is/n)| = 2\sin(\pi k/n)$.

As another example of the application of the eq.(47) we consider the critical Ashkin Teller (AT) model perturbed by the thermal operator. The spin variable σ (σ^+) in this lattice model takes the values in the group Z_4, which determines the symmetry of AT model. This model has the critical line (in two dimensional space of parameters), where it can be described by CFT. This critical line can be parametrized by the conformal dimension $\Delta_\varepsilon = \gamma^2$ of the thermal operator ε. The central charge of the corresponding CFT is equal to one along the whole critical line. This CFT can be described by the action of a massless free scalar field ϕ (See ref. [15] for details). The thermal operator ε can be represented through this field as $\varepsilon = \sqrt{2}\cos(\gamma\phi)$. The action of the AT model perturbed by this operator has the form of Sine-Gordon QFT:

$$\mathcal{A}_{SG} = \int d^2x \left(\frac{(\partial_\mu \phi)^2}{16\pi} + \lambda\sqrt{2}\cos(\gamma\phi). \right) \tag{49}$$

The order parameters in the critical AT model are the spin fields σ (σ^+) and $\Sigma \sim \sigma^2$. The field σ (σ^+) has conformal dimension $\Delta_\sigma = 1/16$ which is independent on γ along the whole critical line. This field is non-local with respect to the field ϕ which has square-root branch point at the position of σ. The field Σ is local with respect to ϕ. It has conformal dimension $\Delta_\Sigma = \gamma^2/4$ and can be represented in terms of ϕ as $\Sigma = \pm\sqrt{2}\cos(\gamma\phi/2)$.

At the points $\gamma^2 = 1/n$ the critical AT model can be described by $\mathcal{M}_{h+1}(D_n)$ minimal models with $\beta^2 = \frac{2n-1}{2n}$ ($u = \frac{1}{2n}$) which have the central charge $c = 1$ for all n. The perturbing operator $\Phi_{ad} \in \mathcal{M}_{h+1}(D_n)$ has the conformal dimension $\Delta_\varepsilon = 1/n = \gamma^2$ and is exactly the thermal operator ε. The operator σ (σ^+) can be represented by the field $\Phi_{\omega_n}(\Phi_{\omega_{n-1}}) \in \mathcal{M}_{h+1}(D_n)$ for odd n and by the field $\exp(i\frac{\pi}{4})(\Phi_{\omega_n} + i\Phi_{\omega_{n-1}})/\sqrt{2}$ for even n. Here ω_n, ω_{n-1} denote the fundamental weights of the two spinor representations of D_n. For all n these fields have the conformal dimension $\Delta_\sigma = 1/16$. The operator Σ with dimension $\gamma^2/4$ can be written as $\Sigma = \Phi_{\omega_1}$, where ω_1 is the fundamental weight of the vector representation.

In the ordered phase the vacua $\{s\}$ in the perturbed CFT (33) are specified by the highest weights θ_s which satisfy the inequality (46). For perturbed $\mathcal{M}_{h+1}(D_n)$ CFTs this condition has four solutions, namely, $\theta_s = \{0, \omega_1, \omega_{n-1}, \omega_n\}$ in agreement with Z_4 symmetry of AT model. The expectation values of the order parameters σ and Σ can be calculated now, using eq.(47). For this calculation it is convenient to use eqs.(31) and (42), which for Lie algebra D_n can be written as:

$$\begin{aligned}
\mathcal{N}^2(-\beta\omega_{n-1}) &= \mathcal{N}^2(-\beta\omega_n) = (\pi\gamma(u))^{n(n-1)/2} \prod_{i=1}^{[n/2]} \frac{\gamma((2i-1)u)}{\gamma((2n-2i)u)}; \\
\mathcal{N}^2(-\beta\omega_1) &= (\pi\gamma(u))^{2(n-1)} \frac{\gamma(u)\gamma((n-1)u)}{\gamma(nu)\gamma(2(n-1)u)};
\end{aligned} \tag{50}$$

and

$$\mathcal{F}_\beta(-\beta\omega_{n-1}) = \mathcal{F}_\beta(-\beta\omega_n) = \frac{\sinh(\beta^2 t)\sinh(nut)\sinh(hut/2)}{\sinh t \sinh(2ut)\sinh(hut)}$$

$$\mathcal{F}_\beta(-\beta\omega_n) = \frac{\sinh(\beta^2 t)\cosh((n-2)ut)}{t\sinh t\cosh(hut/2)}. \tag{51}$$

After simple transformations the vacuum expectation values for the order parameters can be expressed through the soliton mass M in the Sine-Gordon theory and parameter $\gamma^2 = 2u$. Namely, we obtain:

$$\begin{aligned}\langle\sigma\rangle_s &= \exp(i\pi s/2)\left(\frac{2M\sqrt{\pi}\Gamma\left(\frac{1}{2(1-\gamma^2)}\right)}{\Gamma\left(\frac{\gamma^2}{2(1-\gamma^2)}\right)}\right)^{1/8} \\ &\times\exp\left(\int\frac{dt}{8t}\left(\frac{\cosh(2(1-\gamma^2)t)}{\cosh t\cosh\gamma^2 t\cosh((1-\gamma^2)t)} - e^{-2t}\right)\right). \end{aligned}\tag{52}$$

In this form eq.(52) can be generalized to the arbitrary values of parameter $\gamma^2 < 1$. In particular, at $\gamma^2 = 2/3$ the expectation value of field σ was calculated in [16], using different approach. The result is in exact agreement with eq.(52).

The expectation values of the operator $\Sigma = \Phi_{\omega_1}$, calculated from eq.(47) have the form:

$$\begin{aligned}\langle\Sigma\rangle_s &= (-1)^s\sqrt{2}\left(\frac{M\sqrt{\pi}\Gamma\left(\frac{1}{2(1-\gamma^2)}\right)}{2\Gamma\left(\frac{\gamma^2}{2(1-\gamma^2)}\right)}\right)^{\gamma^2/2} \\ &\times\exp\left(\int\frac{dt}{t}\left(\frac{\sinh(\gamma^2 t)}{2\sinh t\cosh((1-\gamma^2)t)} - \frac{\gamma^2}{2}e^{-2t}\right)\right) \\ &= \pm\sqrt{2}\,\langle\cos(\gamma\phi/2)\rangle_{SG} \end{aligned}\tag{53}$$

The last equality shows that this VEV coincides with the expectation value for the field $\pm\sqrt{2}\cos(\gamma\phi)$ in Sine-Gordon model calculated in [1]. It gives us nonperturbative test for the eq.(47).

At the end of this section we note that there are other nonperturbative tests for eq.(47). For example, the CFTs $\mathcal{M}_{h+1}(E_{6,7})$ perturbed by operator Φ_{ad} coincide with tricritical Z_3 Potts and tricritical Ising models perturbed by the first thermal operator. For the case of $G = E_8$ this theory coincide with Ising model in the magnetic field. The same models can be described as the minimal models of CFT, i.e. $\mathcal{M}_p(A_1)$ ($p = 6, 4, 3$) perturbed by the operator Φ_{12}. The expectation values of the fields in the minimal models perturbed by the operator Φ_{12} where calculated in [2], using the quantum restriction of imaginary Bullough-Dodd model. The calculation of the expectation values of the fields in perturbed CFTs $\mathcal{M}_{h+1}(E)$ gives the exact agreement with the results of paper [2].

4 Asymptotics of Cylindrical Solutions of Classical Toda Equations

In this section we consider the application of VEVs $G(a)$ of the exponential fields to the analysis of the special class of the solutions of classical Toda equations, which appears in the topological QFTs [6]. These solutions possess the cylindrical symmetry (i.e. depend only on $r = |x|$):

$$\partial_r^2 \phi + r^{-1} \partial_r \phi = m^2 \sum_{i=0}^{r} n_i e_i \exp(e_i \cdot \phi); \qquad n_0 = 1, \tag{54}$$

and satisfy the following asymptotic conditions:

$$\phi \to -2a \log(mr) + B(a) \qquad at \quad r \to 0; \tag{55}$$

$$\phi \to \sum_{i=1}^{r} \eta_i X_i(a) K_0(\nu_i mr) \qquad at \quad r \to \infty, \tag{56}$$

where $K_0(t)$ is the Mc-Donald function and η_i are the eigenvectors of mass matrix (5): $M\eta_i = \nu_i^2 \eta_i$, satisfying the normalization condition:

$$\eta_{\bar{i}} = \bar{\eta}_i; \quad \bar{\eta}_i \cdot \eta_i = h; \quad \mathrm{Re}\, \eta_i \cdot \rho > 0. \tag{57}$$

The solutions with these properties exist, if vector a satisfies the conditions: $a \cdot e_i < 1$, $i = 0, ...r$. In this case all other terms in the short distance expansion (55) can be determined from the first two using eq.(54). The constant term $B(a)$ in eq.(55) is not arbitrary. It is defined by the global properties of eq.(54). Only for special values of vector B the solution has no singularities and decrease exponentially at infinity. To determine the function $B(a)$ we consider the semiclassical limit ($b \to 0$) of the QFT (1). At this limit we have (see eq.(6)) that $\mu = (mk(G))^2/4\pi b^2 + O(1)$ and after rescaling of field $\varphi: \ b\varphi = \tilde{\varphi}$ the action (1) is proportional to $1/b^2$:

$$\mathcal{A}_b = \frac{1}{4\pi b^2} \int d^2x \left[\frac{1}{2} (\partial_\mu \tilde{\varphi})^2 + (mk(G))^2 \sum_{i=0}^{r} e^{e_i \cdot \tilde{\varphi}} \right] + O(1). \tag{58}$$

The classical equations with cylindrical symmetry following from this action coincide with eqs.(54) after the shift: $\phi = \tilde{\varphi} - \tilde{\varphi}_0$, where constant term $\tilde{\varphi}_0$ corresponds to the classical vacuum of ATT (1). It can be written in terms of the fundamental weights ω_i as:

$$\tilde{\varphi}_0 = b\varphi_0 = \sum_{i=1}^{r} (\log n_i - 2 \log k(G)) \omega_i. \tag{59}$$

It is easy to see now that solution ϕ to eq.(54) can be expressed through the semiclassical limit of the following two point function (which is completely determined by the saddle point contribution):

$$\phi + \tilde{\varphi}_0 = \lim_{b\to 0}\left(\frac{\langle b\varphi(x)\exp(a\cdot\varphi(0)/b)\rangle_b}{G(a/b)}\right). \tag{60}$$

The asymptotics (55) is governed by the exponential term in the correlation function and asymptotics (56) follows from form-factor expansion for (60). The vector $B(a)$ can be now derived from the operator product expansion:

$$\frac{\left\langle b\varphi(x)e^{a\cdot\varphi(0)/b}\right\rangle_b}{G(a/b)} = -2a\log r + \frac{\left\langle b\varphi(0)e^{a\cdot\varphi(0)/b}\right\rangle_b}{G(a/b)} + O(r^\sigma). \tag{61}$$

It means that:

$$\phi + \tilde{\varphi}_0 = -2a\log r + \lim_{b\to 0} b^2\partial_a \log G(a/b) + O(r^\sigma); \quad \sigma > 0. \tag{62}$$

This limit can be calculated explicitly with the result:

$$\phi + \tilde{\varphi}_0 = -2a\log\left(\frac{mrk(G)}{2h}\right) - \sum_{\alpha>0}\alpha\log\gamma\left(\frac{(\rho-a)_\alpha}{h}\right) + O(r^\sigma). \tag{63}$$

At $a = 0$ the solution ϕ vanishes, so we have two different expressions for $\tilde{\varphi}_0$, namely, eq.(63) at $a = 0$ and eq.(59). Comparing them we obtain the amusing relation for gamma-functions:

$$\prod_{\alpha>0}\left(\gamma\left(\frac{\rho_\alpha}{h}\right)\right)^{-\alpha\cdot e_j} = n_j\,(k(G))^{-2}, \tag{64}$$

which is valid for all simply laced Lie algebras. As the final expression for the constant term in eq.(55) we have:

$$B(a) = -2a\log\left(\frac{k(G)}{2h}\right) - \sum_{\alpha>0}\alpha\left(\log\gamma\left(\frac{(\rho-a)_\alpha}{h}\right) - \log\gamma\left(\frac{\rho_\alpha}{h}\right)\right). \tag{65}$$

The same expression for the asymptotics of A_n- Toda solutions was obtained by completely different method in ref.[17].

The long distance asymptotics (56) can be derived from the one particle form-factors in ATT (see ref.[18] for details). At the semiclassical limit $b \to 0$ these form-factors are related with the coefficients $X_i(a)$ as:

$$\langle 0 \left|\exp(a\cdot\varphi/b)\right| A_i\rangle = G(a/b)\left(\frac{h}{2\pi}\right)^{1/2} X_i(a)\left(1+O(b^2)\right). \tag{66}$$

It can be derived from form-factor equations that functions $X_i(a)$ are equal to the characters of the fundamental representations $Y_i(G)$ of the Yangian $Y(G)$:

$$X_i(a) = Tr_{Y_i}\exp\left(\frac{2\pi i}{h}(a-\rho)\cdot H\right). \tag{67}$$

The functions $X_i(a)$ can be expressed through the similar characters $\chi_i(a)$ taken over the fundamental representations $\pi_i(G)$ of Lie algebra G. For all i that satisfy the condition: $-e_0\cdot\omega_i = 1$ functions $X_i(a)$ coincide with characters $\chi_i(a)$. For A_n this is valid for all i. For D_n we have that: $X_1 = \chi_1, X_{n-1} = \chi_{n-1}, X_n = \chi_n$, and for other representations:

$$X_{2j}(a) = 1+\sum_{s=1}^{j}\chi_{2s}(a); \quad X_{2j+1}(a) = \sum_{s=0}^{j}\chi_{2s+1}(a). \tag{68}$$

For the fundamental representations of D and E that coincide with adjoint the functions $X_{ad}(a)$ are equal to $1+\chi_{ad}(a)$. The list of expressions for functions X_i in terms of the characters χ_i can be found in [19]. It can be shown that functions $X_i(a)$ possess the properties:

$$X_i(0) = 0, \quad \frac{h}{2\pi}\partial_a X_i(0) = \eta_i; \quad X_i\left(\frac{\rho}{h+1}\right) = 1, \tag{69}$$

where η_i are the eigenvectors of mass matrix (5) satisfying normalization conditions (57).

At the special points $a = \rho/(h+1)$ all X_i are equal to one. At these points the short distance expansion for the solution ϕ also simplifies drastically (all higher terms in the asymptotics (55) are given by the regular series in $(mr)^{2/(h+1)}$). These points appear in connection with special class of topological QFT. In particular, the solutions of eq.(54) for these values of a describe "the new sypersymmetric index" [6] for the imaginary ATT at $N=2$ sypersymmetric points: $\beta^2 = h/(h+1)$ [20]. We note that ATT give us an example of the system where representations of Yangian appear already in the classical case. We suppose to return to the analysis of eq. (54) in the separate publication.

5 Boundary Toda Theories, Reflection Amplitudes, One Point Functions and Ground State Energies

At the previous sections we considered Toda theories defined in the whole plane R^2. Here we consider simply-laced NATT and ATT defined at the half-plane $H = (x, y; y > 0)$ with the integrable boundary conditions.

The integrability conditions for the classical simply-laced ATT on H were studied in the paper[22]. It was shown there that the action of integrable ATT can be written as:

$$\begin{aligned}\mathcal{A}_{bound} &= \int_H d^2x \left[\frac{1}{8\pi}(\partial_\mu\varphi)^2 + \mu\sum_{i=1}^{r} e^{be_i\cdot\varphi} + \mu e^{be_0\cdot\varphi}\right] \\ &+\mu_B \int dx \sum_0^r d_i e^{be_i\cdot\varphi/2} \end{aligned} \tag{70}$$

where or all the parameters $d_i = 0$, that corresponds to the Neumann boundary conditions:

$$\partial_y\varphi(x, 0) = 0; \tag{71}$$

or the parameters $d_i = \pm 1$ and the parameter μ_B is related with the parameter μ in the bulk (in classical case) as:

$$\mu_B^2 = \mu/\pi b^2. \tag{72}$$

For Lie algebra A_1 (Sinh-Gordon model) the integrability conditions are much less restrictive and parameters d_0 and d_1 can have arbitrary values [21]. The background CFT for this case is the boundary Liouville theory. The reflection amplitudes in boundary Liouville CFT for arbitrary values of parameter μ_B or (d_1) and VEVs in boundary Sinh-Gordon model for arbitrary values of d_1 and d_0 were found in [25] . Here we discuss this problem for other Lie algebras where the choice of integrable conditions is rather restrictive. We consider the Toda theories with Neumann boundary conditions and in the case when all parameter $d_i = 1$ (with $d_0 = 0$ in non-affine case). Really, these two quite different classical theories in the quantum case are described by the same quantum theory and are related by duality transformation $(b \to 1/b)$ [23], [24]. The cases corresponding to different signs of parameters d_i is more subtle and will be considered elsewhere.

We start from the consideration of the boundary NATTs which are described by the action (70) without the last term in the bulk action and $d_0 = 0$

in the boundary term. The boundary ATTs will be considered as perturbed boundary CFTs. At the whole plane NATTs possess the infinite symmetry generated by two copies of chiral $W(G)-$ algebras. These $W(G)$-algebras contain r holomorphic and r antyholomorphic currents $W_j(z)$ and $\overline{W}_j(\overline{z})$ with spins that follow the exponents of Lie algebra G. At the half-plane with W-invariant boundary conditions we have only one W-algebra. In this case the currents $\overline{W}_j(\overline{z})$ should be the analytical continuations of the currents $W_j(z)$ to the lower half-plane. In particular, they should coincide at the boundary. These conditions impose very strong restrictions to the form of the boundary terms in the action. It is rather easy to derive from the explicit form of W-currents [9] that Neumann boundary conditions (71) preserve W-symmetry. The boundary condition (72) whose quantum modified version has a form [25]:

$$\mu_B^2 = \frac{\mu}{2}\cot\left(\frac{\pi b^2}{2}\right) \tag{73}$$

with all $d_i = 1$ and $d_0 = 0$ describes the dual theory and, hence, also preserve $W-$symmetry.

In the boundary Liouville CFT [25] the equation (73) was obtained as the condition that boundary exponential fields corresponding to the degenerate representations of Virasoro algebra satisfy the null vectors equations. It can be shown that the same condition (73) is valid for boundary exponential fields in NATT corresponding to the strongly degenerate representations of the W-algebra. The null vectors equations simplify drastically the OPE with strongly degenerate fields. We will use this property for the calculation of the boundary reflection amplitudes. We note that in boundary Liouville theory the condition that null vectors equations for degenerate boundary fields are fulfiled is not nesessary for conformal invariance and here it is, probably, necessary for preserving of W-symmetry.

In the W-invariant boundary NATT we have two kinds of the exponential fields. The bulk fields $V_a(x,y)$ and the boundary fields $B_a(x)$ defined as:

$$V_a(x,y) = \exp\{a\cdot\varphi(x,y)\}; \quad B_a(x) = \exp\{a\cdot\varphi(x)/2\}. \tag{74}$$

These fields are specified by the same r eigenvalues $w_j(a)$ that and corresponding fields (12) defined on the whole plane. In particular, their dimensions are given by eq.(13). The functions $w_j(a)$ are invariant under the action of the Weyl group of G (see section 2), defined by eq.(14) and, hence, we can introduce the boundary reflection amplitudes $\mathcal{R}_s(a)$ as:

$$B_a(x) = \mathcal{R}_s(a)B_{s(a)}(x). \tag{75}$$

The reflection amplitudes can be easily expressed through the two point functions of boundary fields:

$$D(a) = \langle B_a(0), B_a(x)\rangle\, |x|^{2\Delta(a)}. \tag{76}$$

For the evaluation of two point functions we can use following the lines of section 2 the screening charges which commute with all generators of W -symmetry. In boundary NATT there are two types of screening charges $\widehat{Q}_H(i)$ and $\widehat{Q}_B(i)$:

$$\widehat{Q}_H(i) = \mu \int\limits_H d^2x V_{be_i}(x,y); \quad \widehat{Q}_B(i) = \mu_B \int dx B_{be_i}(x), \tag{77}$$

where μ_B is given by eq.(73).

Using these screening charges we can express the structure constants of the OPE of fields B_a with strongly degenerate boundary fields in terms of Coulomb gas integrals (see ref.[25] for details). The OPEs of fields B_a with these fields contain only the finite number of primary fields. The simplest strongly degenerate fields in NATT are the fields $B_{-b\omega_i}$, where ω_i are the fundamental weights of G. For the calculation of boundary two point functions $D(a)$ it is convenient, following [26], [25], to consider the auxiliary three point functions, including these fields:

$$\langle B_a(x_1) B_{a+b\omega_i}(x_2) B_{-b\omega_i}(x)\rangle\,. \tag{78}$$

Then, tending $x \to x_2$ we can express the asymptotics in terms of function $D(a)$ multiplied by the structure constant, which in the usual normalization is equal to one. Instead, tending $x \to x_1$ we can express the asymptotics in terms of function $D(a+b\omega_i)$ multiplied by the structure constant $\mathbf{C}^{a+b\omega_i}_{a,-b\omega_i}$, which can be calculated using the screening charges (77). Equating these two expressions we obtain:

$$\frac{D(a)}{D(a+b\omega_i)} = \mathbf{C}^{a+b\omega_i}_{a,-b\omega_i} \tag{79}$$

where:

$$\begin{aligned}\mathbf{C}^{a+b\omega_i}_{a,-b\omega_i} \quad = \quad & \left[\frac{\pi\mu 2^{-2-2b^2}}{-\gamma(-b^2)}\right]^{\omega_i\cdot\rho} \prod_{\alpha>0}\prod_{k=1}^{\alpha\cdot\omega_i} \frac{\Gamma((b\overline{a}_\alpha + (k-1)b^2)/2)}{\Gamma((1-b\overline{a}_\alpha - (k-1)b^2)/2)} \\ & \times \frac{\Gamma(-(b\overline{a}_\alpha + kb^2)/2)}{\Gamma((1+b\overline{a}_\alpha + kb^2)/2)}\end{aligned} \tag{80}$$

here $\bar{a} = (a-Q), \bar{a}_\alpha = (a-Q)\cdot\alpha$ like and in eq.(23).

To construct a solution to these functional equations it is convenient to use the special function $\mathbf{G}(x)$ (see for example [25]), which is self dual entire function with zeroes at $x = -nb - m/b; n, m = 0, 1, 2...$ and enjoys the following shift relations:

$$\mathbf{G}(x+b) = \frac{b^{1/2-bx}}{\sqrt{2\pi}}\Gamma(bx)\mathbf{G}(x); \qquad \mathbf{G}(x+1/b) = \frac{b^{x/b-1/2}}{\sqrt{2\pi}}\Gamma(x/b)\mathbf{G}(x). \quad (81)$$

The integral representation which is valid for $\mathrm{Re}\, x > 0$, has a form:

$$\log \mathbf{G}(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-qt/2} - e^{-xt}}{(1-e^{-bt})(1-e^{t/b})} + \frac{(q/2-x)^2}{2}e^{-t} + \frac{q/2-x}{t}\right] \quad (82)$$

where:

$$q = b + 1/b. \quad (83)$$

With this function the solution to the functional equations (79,80), satisfying the normalization condition $D(a)D(2Q-a) = 1$ can be written in the form:

$$D(a) = \frac{A_B(2Q-a)}{A_B(a)} \quad (84)$$

where:

$$\begin{aligned} A_B(a) &= \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\bar{a}\cdot\rho/2b} \\ &\times \prod_{\alpha>0} \frac{\mathbf{G}(\bar{a}_\alpha)\mathbf{G}^2((q-\bar{a}_\alpha)/2)}{\mathbf{G}((q+b+\bar{a}_\alpha)/2)\mathbf{G}((q-b+\bar{a}_\alpha)/2)}. \end{aligned} \quad (85)$$

Reflection amplitude for arbitrary element $\hat{\mathbf{s}}$ of the Weyl group of Lie algebra G can be written as:

$$\mathcal{R}_s(a) = \frac{A_B(s(a))}{A_B(a)} = \frac{A_B(Q + \hat{\mathbf{s}}(a-Q))}{A_B(a)}. \quad (86)$$

These equations describe two point functions and reflection amplitudes for NATT with boundary conditions (73). To obtain the same values for the dual theory, which corresponds to the Neumann boundary conditions (71) we should change in eq.(85) $b \to 1/b$ and transform the bulk parameter $\mu \to \tilde{\mu}$, where $\pi\mu\gamma(b^2) = (\pi\tilde{\mu}\gamma(1/b^2))^{b^2}$. We note that unlike boundary reflection amplitudes the bulk reflection amplitudes (24,25) are invariant under this transformation.

Boundary reflection amplitudes (85,86) can be used for the calculation of vacuum expectation values of the boundary exponential fields in ATTs, in the same way as it was done for in section 3 for ATTs defined at the whole plane. Here we adopt the conventual normalization of boundary exponential fields (see e.g. [2]) corresponding to the short distance asymptotics at $|x_1 - x_2| \to 0$:

$$e^{a\varphi/2}(x_1)e^{a\varphi/2}(x_2) = |x_1 - x_2|^{a^2} + ...$$

We define boundary one point function $G_B(a)$ as:

$$G_B(a) = \langle \exp(a \cdot \varphi/2) \rangle_B \tag{87}$$

For any element $\widehat{\mathbf{s}}$ of Weyl group $\mathcal{W}$ this function satisfies the functional reflection relation:

$$G_B(a) = \mathcal{R}_s(a) G_B(s(a)). \tag{88}$$

The minimal meromorphic solution to these functional equations which respects all the symmetries of extended Dynkin diagrams of Lie algebras ADE has the form:

$$\begin{aligned} G_B(a) &= \left[\frac{mk(G)\Gamma\left(\frac{1}{(1+b^2)h}\right)\Gamma\left(\frac{b^2}{(1+b^2)h}\right)b^2}{2\Gamma\left(\frac{1}{h}\right)h(1+b^2)}\right]^{-a^2/2} \\ &\times \exp\left(\int \frac{dt}{t}[\frac{a^2}{2}e^{-2t} - F_B(a,t)]\right) \end{aligned} \tag{89}$$

with

$$F_B(a,t) = f(t) \sum_{\alpha>0} \frac{\sinh(ba_\alpha t)\sinh((b(a-2Q)_\alpha + h(1+b^2))t)}{\sinh 2t \sinh(2b^2 t)\sinh((1+b^2)ht)} \tag{90}$$

where for boundary conditions (73) function $f(t)$ is:

$$f(t) = 2e^t \sinh((1+b^2)t)\cosh(b^2 t) \tag{91}$$

and for dual theory which corresponds to Neumann boundary conditions (71) we should do the substitution $f(t) \to \widetilde{f}(t)$:

$$\widetilde{f}(t) = 2e^{tb^2}\sinh((1+b^2)t)\cosh t. \tag{92}$$

It is easy to see from the explicit form of $G_B(a)$ that in the classical limit ($b \to 0, b\varphi$ is fixed) the boundary VEV $\widetilde{\varphi}_{0,B}$ of the field $b\varphi$ for the Neumann boundary conditions coincides with classical vacuum $\widetilde{\varphi}_0$ (59) in the bulk. For the boundary conditions (73) it happens only for Lie algebra A_r, where both these values vanish. For other cases we can derive from eqs.(89-91) that

$$\widetilde{\varphi}_{0,B} = \widetilde{\varphi}_0 + \vartheta \tag{93}$$

where:

$$\vartheta = -\sum_{\alpha>0} \alpha \int \frac{dt}{t} \frac{\sinh((h-2\rho_\alpha)t)}{\sinh(ht)} \tanh t. \tag{94}$$

These integrals can be calculated explicitly and expressed in terms of the logarithms of the trigonometric functions of the parameter π/h. Vector ϑ is simply related with boundary soliton solution which describes the classical vacuum configuration. The classical problem for this solution $\phi(y), y > 0$ can be formulated in the following way. We are looking for the solution to classical Toda equations, which decrease at $y \to \infty$ and satisfies at $y = 0$ the boundary conditions that follow from action (70). After rescaling and shifting (see section 4) the field $\phi = b\varphi - \widetilde{\varphi}_0$ satisfies the equation:

$$\partial_y^2 \phi = m^2 \sum_{i=0}^{r} n_i e_i \exp(e_i \cdot \phi); \qquad n_0 = 1 \tag{95}$$

and boundary condition at $y = 0$:

$$\partial_y \phi = m \sum_{i=0}^{r} \sqrt{n_i} e_i \exp(e_i \cdot \phi/2). \tag{96}$$

The vector ϑ is equal to the boundary value of this solution: $\vartheta = \phi(0)$, and hence, it completely fixes the solution.

We note that eq.(95) has r independent integrals. These integrals give the equations to parameter ϑ. In particular, it is easy to derive from the first integral that numbers $E_i = \exp(e_i \cdot \vartheta/2), i = 0, 1..., r$ should satisfy the following "sum rules":

$$\sum_{i=0}^{r} \sum_{j=0}^{r} (2\delta_{ij} - e_i \cdot e_j) \sqrt{n_i} \sqrt{n_j} E_i E_j = 2h. \tag{97}$$

Numbers E_i possess all the symmetries of extended Dynkin diagram of Lie algebra G. Together with these symmetries the "sum rules" fix completely parameters E_i (and hence vector ϑ) for Lie algebras D_4 and D_5. Consider, for example, the second case. There we have that $E_0 = E_1 = E_4 = E_5 = u$, and $E_2 = E_3 = v$. But numbers E_i are not independent. They satisfy the condition: $E_0^{-1} = E_1 E_4 E_5 E_2^2 E_3^2$, or $u = 1/v$. Then we find from eq.(97) that

$4\sqrt{2} + 2v^2 = 8$. This defines all parameters E_i. The result is in perfect agreement with eq.(94), which for Lie algebra D_r can be rewritten as:

$$E_0^2 = E_1^2 = E_{r-1}^2 = E_r^2 = \frac{8}{h^2 \sin^2(\pi/h)};$$

$$E_k^2 = \exp\left(\int \frac{dx}{x} \frac{4\sinh^2 x \cosh 2(r-2k)x}{\sinh 2(r-1)x \cosh 2x} \right), \quad k = 2, ..., r-2. \tag{98}$$

For other Lie algebras we can only check that vector ϑ defined by eq.(94) satisfies eq. (97).

The solution to the eqs. (95),(96) can be expressed in terms of tau-functions associated with multi-soliton solutions of classical ATT equations (see, for example [27],[28]). For D and E series of algebras (besides the cases D_4 and D_5 [27]) the explicit form of these solutions is not known. We suppose to discuss these solutions in more details in the separate publication. Here we consider the classical boundary ground state energy which can be defined as:

$$\mathcal{E}_{bound}^{(cl)} = \frac{1}{4\pi b^2}[2m \sum_{i=0}^{r} \sqrt{n_i} E_i + \int_0^\infty dy (\frac{1}{2}(\partial_y \phi)^2 + m^2 \sum_{i=0}^{r} n_i e^{e_i \cdot \phi})]. \tag{99}$$

We note that numerical values of ϑ^2, defined by eq.(94) are rather small for all G, and the integral term in eq.(99) can be calculated with the good accuracy using the bilinear approximation:

$$\int_0^\infty dy (\frac{1}{2}(\partial_y \phi)^2 + m^2 \sum_{i=0}^{r} n_i e^{e_i \cdot \phi}) = \frac{m}{2} \sum_{a,b} (M^{1/2})_{ab} \vartheta_a \vartheta_b + O(|\vartheta|^3)$$

where M_{ab} is the mass matrix of ATT, defined by eq.(5).

More careful analysis of the eqs.(95),(96) gives us the reasons to write the expression for boundary ground state energy in the following form. Namely, we denote as $\Sigma_m(G)$ the sum of masses of all particles in ATT:

$$\Sigma_m(G) = \sum_{i=1}^{r} m_i = m \sum_{i=1}^{r} \nu_i = m \cdot tr(M^{1/2}). \tag{100}$$

It has the following values for simply-laced Lie algebras:

$$
\begin{aligned}
\Sigma_m(A_n) &= 2m\cot(\pi/2h); \quad \Sigma_m(D_n) = \frac{2m\cos(\pi/4-\pi/2h)}{\sin(\pi/2h)}; \\
\Sigma_m(E_6) &= m(6-2\sqrt{3})^{1/2}\frac{\cos(\pi/8)}{\sin(\pi/2h)}; \quad \Sigma_m(E_7) = \frac{2m\sin(2\pi/9)}{\sin(\pi/2h)}; \\
\Sigma_m(E_8) &= 4m(\sqrt{3}\sin(\pi/5)\sin(\pi/30))^{1/2}\frac{\cos(\pi/5)}{\sin(\pi/2h)}.
\end{aligned} \tag{101}
$$

The classical boundary ground state energy (99) can be written in terms of these values as:

$$
\mathcal{E}^{(cl)}_{bound}(G) = \frac{h}{4\pi b^2}\tan(\pi/2h)\Sigma_m(G). \tag{102}
$$

In quantum case the boundary ground state energy $\mathcal{E}^{(q)}_{bound}$ will have the contributions coming from the boundary term in the Hamiltonian and from the bulk fluctuations around the background solution. The contributions of the first type can be calculated using the explicit expression for vacuum expectation values (89). For small b the first quantum correction of the second type can be expressed through the boundary $S-$matrix at $b=0$ (see, for example [29],[30]). We will discuss these boundary S-matrices bellow. Here we note that at the strong coupling region $b >> 1$ our theory is described by the weakly coupled dual ATT with Neumann boundary conditions (71). At strong coupling limit the dual theory is the set of r free bosonic theories with masses m_i. The boundary ground state energy for free massive bosonic theory with Neumann boundary conditions and mass m_i can be easily calculated and is equal to $m_i/8$. The first perturbative correction in the weakly coupled dual theory can be also evaluated with the result:

$$
\mathcal{E}^{(q)}_{bound}(G) = \frac{\Sigma_m(G)}{8}\left(1+\frac{\pi}{2hb^2}\cot(\pi/2h)+O(1/b^4)\right). \tag{103}
$$

Both asymptotics $b\to 0$ (102) and $b\to\infty$ (103) are in agreement with the following conjecture for boundary ground state energy:

$$
\mathcal{E}^{(q)}_{bound}(G) = \frac{\sin(\pi/2h)\Sigma_m(G)}{8\sin(\pi x/2h)\cos(\pi(1-x)/2h)} \tag{104}
$$

where:

$$
x = \frac{b^2}{1+b^2} = \frac{B}{2} = \frac{b}{q}. \tag{105}
$$

The nonperturbative check of this conjecture can be done using the boundary Thermodynamic Bethe Ansatz equations [31]. The kernels in these nonlinear integral equations depend on the bulk and boundary S-matrices. The boundary S-matrix (reflection coefficient) in ATT for the particle j corresponding to the fundamental representation $\pi_j(G)$ can be defined as:

$$|j,-\theta\rangle_{out} = K_j(\theta)|j,\theta\rangle_{in} \tag{106}$$

where θ is the rapidity of particle j.

The reflection coefficients for A_{n-1} ATT with boundary conditions discussed above were conjectured in [32],[33]. They can be written in terms of function:

$$(z) = \frac{\sin\left(\frac{\theta}{2i} + \frac{\pi}{2h}z\right)}{\sin\left(\frac{\theta}{2i} - \frac{\pi}{2h}z\right)} \tag{107}$$

in the following way:

$$K_j(\theta) = \prod_{a=1}^{j}(a-1)(a-n)(-a+x)(-a-n+1-x). \tag{108}$$

Unfortunately, we were not able to find in the literature the conjecture for other Lie algebras (may be it exist) consistent with duality properties discussed above. So, we will give here the conjecture which naturally generalizes the reflection coefficients (108) to the other simply-laced Lie algebras. To do this we rewrite eq. (108) in the form:

$$K_j(\theta) = \exp(-i\delta_j(\theta)) \tag{109}$$

where

$$\delta_j = \int \frac{dt}{t}\sinh(\frac{2h\theta t}{\pi})[\sinh((1-x)t)\sinh((h+x)t)\Delta_j(A_{n-1},t) - 2] \tag{110}$$

and

$$\Delta_j(A_{n-1},t) = \frac{8\sinh(jt)\sinh((n-j)t)}{\sinh(t)\sinh(2ht)}. \tag{111}$$

The natural generalization of these equations can be written as:

$$K_j(\theta) = \Phi_j(\theta)\exp(-i\delta_j(\theta)) \tag{112}$$

where $\Phi_j(\theta)$ are CDD factors, satisfying the conditions:

$$\Phi_j(\theta)\Phi_j(-\theta) = 1; \quad \Phi(\theta)\Phi(\theta+i\pi) = 1 \tag{113}$$

and function $\delta_j(\theta)$ is defined by eq.(110) with the substitution $\Delta_j(A_{n-1}) \to \Delta_j(G,t)$, where:

$$\Delta_j(G,t) = \frac{4}{\cosh(ht)}\left((2\cosh(t)-2)\delta_{mn} + e_m \cdot e_n\right)^{-1}_{jj}. \tag{114}$$

The most important part of this conjecture is that for the particles j corresponding to fundamental representations $\pi_j(G)$ with the fundamental weights satisfying the condition $-e_0 \cdot \omega_j = 1$ (or n_j in eq.(2) is equal to 1), the CDD factors $\Phi_j(\theta)$ in eq.(112) are equal to one.

This statement fixes completely the boundary S-matrix for Lie algebras D_n, E_6, E_7. We denote as $K_f(G,\theta)$ the fundamental reflection coefficients. It means that all other amplitudes $K_j(\theta)$ can be obtained from fundamental reflection factors $K_f(G,\theta)$ by the application of boundary bootstrap fusion procedure [21],[34]. It is easy to see that $K_f(D_n,\theta) = K_n(\theta) = K_{n-1}(\theta)$, where particles n and $n-1$ correspond to the spinor representations of D_n; $K_f(E_6,\theta) = K_1(\theta) = K_{\overline{1}}(\theta)$, where particles 1 and $\overline{1}$ form the doublet of lightest particles in E_6 ATT and $K_f(E_7,\theta) = K_1(\theta)$, where 1 is the lightest particle in E_7 ATT. For all these three cases the CDD factors $\Phi_f(\theta) = 1$ and the reflection coefficients are given by eqs.(109)(110) with $j = f$ and

$$\Delta_f(G,t) = \frac{8\sinh(ht/2)\sinh((h/2+q-1)t)}{\sinh(qt)\sinh(2ht)} \tag{115}$$

where $q(G) = \max_i n_i; q(D) = 2, q(E_6) = 3, q(E_7) = 4$.

As an example of the application of boundary bootstrap equations we give here the reflection coefficients K_j for the particles $j = 1, 2, ..., n-2$ in D_n ATT, which can be obtained from the amplitude $K_f(\theta) = K_n(\theta) = K_{n-1}(\theta)$. These functions can be written in the form (112), where:

$$\Delta_j(D_n,t) = \frac{16\sinh jt\cosh((h/2-j)t)\sinh(ht/2)}{\sinh t\sinh 2ht}$$

and CDD factor

$$\Phi_j = \exp\left(i\int dt\frac{8\sinh(2h\theta t/\pi)\sinh((1-x)t)\cosh xt\sinh((j-1)t)\sinh jt}{t\sinh 2t\cosh ht}\right)$$

Lie algebra E_8 has no fundamental representations with $n_j = 1$. The lightest particle in this case is associated with adjoint representation and $K_f(E_8,\theta) = K_{ad}(\theta)$. The adjoint representations for Lie algebras D, E have $n_{ad} = 2$ and, hence, CDD factors should appear. The reflection coefficients $K_{ad}(\theta)$ for Lie algebras D and E can be written in the the form (112), with:

$$\Delta_{ad}(G,t) = \frac{8\cosh((q-1)t)\cosh((h/2-q)t)}{\cosh(ht/2)\cosh ht}$$

where $q(G)$ is defined above ($q(E_8) = 6$) and CDD factor is:

$$\Phi_{ad}(\theta) = \exp\left(i \int dt \frac{8\sinh(2h\theta t/\pi)\sinh((1-x)t)\cosh xt \sinh t}{t\cosh ht}\right).$$

The analysis of the boundary Thermodynamic Bethe Ansatz equations with kernels depending on the reflection coefficients written above gives the exact agreement with eq.(104) for the boundary ground state energy.

At the end of this section we note that semiclassical limit of function $G_B(a)$ contains the information about the short distance asymptotics for the following classical boundary problem. Let $\phi(x, y)$ be a smooth function that for $y > 0$ solves the equations:

$$(\partial_y^2 + \partial_x^2)\phi = m^2 \sum_{i=0}^{r} n_i e_i \exp(e_i \cdot \phi); \quad \partial_y \phi(x, 0) = m \sum_{i=0}^{r} \sqrt{n_i} e_i \exp(e_i \cdot \phi/2)$$

and satisfies the following asymptotic conditions:

$$\phi \to 0, \quad x^2 + y^2 \to \infty; \quad \phi \to -a\log(x^2 + y^2) + \mathcal{B}(a), \quad x^2 + y^2 \to 0. \tag{116}$$

Then, exactly at the same way as it was done in section 4, we can derive that $\mathcal{B}(a) = C(a) - \widetilde{\varphi}_0(0)$, where:

$$C(a) = \lim_{b\to 0} 2b^2 \partial_a \log G_B(a/b). \tag{117}$$

For Neumann boundary conditions (92) we obtain that $\mathcal{B}(a) = B(a)$, where $B(a)$ is given by eq.(65). It is in agreement with a fact that cylindrical solutions, studied in section 4, satisfy the Neumann boundary conditions.

6 Integrable Deformations of Toda Theories and Duality

The duality plays an important role in the analysis of statistical and quantum field theory (QFT) systems. It maps a weak coupling region of one theory to a strong coupling region of the other and makes it possible to use perturbative and semiclassical methods for the study of dual systems in different regions of the coupling constants. For example, a well known duality between sine-Gordon and massive Thirring model [35] together with integrability plays a crucial role for the justification of the exact S-matrix for these QFT . Another interesting example of the duality in two dimensional integrable systems is the weak coupling-strong coupling flow from the affine Toda theories to the same theories with the dual affine Lie algebra [36]. The example of duality in the boundary Toda theories was considered in section 5. The phenomenon of electric-magnetic duality in four dimensional gauge theories conjectured in [37] and developed in [38] opens the possibility for the nonperturbative analysis of the spectrum and the phase structure in the supersymmetric Yang-Mills theories.

Known for many years the phenomenon of duality in QFT still looks rather mysterious and needs further analysis. This analysis essentially simplifies for the 2-d integrable relativistic theories. These QFTs besides the Lagrangian formulation possess also the unambiguous definition in terms of factorized scattering theories (FST). The FST, i.e. the explicit description of the spectrum of particles and their scattering amplitudes, contains all the information about the QFT. These data permit one to use nonperturbative methods for the calculation of the observables in the integrable theories. The comparison of the observables calculated from the FST data and from perturbative or semiclassical analysis based on the Lagrangian formulation makes it possible in some cases to justify the existence of two different (dual) representations for the Lagrangian description of the theory.

In the previous sections we considered simply laced Toda theories which are self-dual. In this section we briefly discuss the massive and conformal field theories, which can be considered as integrable deformations of affine and non-affine Toda theories. We describe duality properties of these theories and calculate the reflection amplitudes for the conformal case. These QFTs for massive case where introduced and studied in ([39]). They form three series of QFTs, numerated by index $\sigma = 1, 2, 3$ (we reserve $\sigma = 0$ for unperturbed CFTs). These QFTs can be described by the scalar field Φ and r-component field $\varphi = (\varphi_{1,\ldots,}\varphi_r)$ with the action:

$$\mathcal{A}_r^{(\sigma)} = \int d^2x \left[\frac{(\partial_\mu \Phi)^2}{8\pi} + \frac{(\partial_\mu \varphi)^2}{8\pi} + 2\mu' \cos(\gamma\Phi) e^{be_r\cdot\varphi} + \mu \sum_{i=1}^{r-1} e^{be_i\cdot\varphi} + U_\sigma(\varphi) \right] \tag{118}$$

where e_i are the simple roots ($e_i \cdot \varphi = \varphi_i - \varphi_{i+1}, i \leq r-1; e_r \cdot \varphi = \varphi_r$) of Lie algebra B_r and the parameters b and γ satisfy the relation:

$$\gamma^2 - b^2 = 1. \tag{119}$$

For these values of parameters the QFT (118) is integrable for three different perturbations U_σ of the CFT, corresponding to $U_\sigma = 0$. Namely:

$$U_1(\varphi) = \mu_1 e^{-2b\varphi_1}; \quad U_2(\varphi) = \mu_2 e^{-b\varphi_1} \quad U_3(\varphi) = \mu_3 e^{-b(\varphi_1+\varphi_2)}. \tag{120}$$

The integrability of these QFTs, was proved in [39] by explicit construction of nontrivial quantum integrals. It is interesting to note that corresponding classical theory is not integrable. The reason is that due to condition (119) the coupling constant γ is not small and we can not reach the classical limit. To understand this limit and to do the QFTs (118) suitable for the prturbative analysis it is convenient to use 2D fermion-boson correspondence [35] and to rewrite the action (118) in the form of massive Thirring model coupled with ATT:

$$\begin{aligned} \mathcal{A}_r^{(\sigma)} &= \int d^2x[\overline{\psi}i\gamma_\mu\partial_\mu\psi - \frac{b^2}{2(1+b^2)}(\overline{\psi}\gamma_\mu\psi)^2 + \pi\mu'\overline{\psi}\psi e^{be_r\cdot\varphi} \\ &+\frac{1}{8\pi}(\partial_\mu\varphi)^2 + \frac{\pi\mu'^2}{4b^2}e^{2be_r\cdot\varphi} + \mu\sum_{i=1}^{r-1}e^{be_i\cdot\varphi} + U_\sigma(\varphi)]. \end{aligned} \tag{121}$$

We added to the action the exponential term $\pi\mu'^2/4b^2e^{2be_r\cdot\varphi}$ as the usual contact contact counterterm to cancel the divergencies coming from fermion loop, however, this term becomes important in the weak coupling (semi-classical) limit. Near the classical vacuum of the QFT (121) the parameter $\mu\sim\mu'^2/b^2$ (the same is true for parameters μ_σ in U_σ) and we can neglect the fermionic terms in the action which do not contain the derivatives. The first term with derivatives can be again bosonized, but it completely decouples in this limit. The classical part of the action is now described by non-simply lased ATT which is, of course, integrable. The corresponding background semiclassical CFT ($U_\sigma = 0$) is described by NATT with Lie algebra C_r.

Integrability imposes strong limitation to the scattering amplitudes (Yang Baxter equations) and permits us to fix completely S-matrix of the QFTs (118,121). Perturbative calculations, analysis of the FST and the Bethe ansatz technique were used in ([39]) to show that these field theories possess the dual representation available for the perturbative analysis in the strong coupling limit, when $\widetilde{b} = 1/b \to 0$. The dual theory can be formulated as the nonlinear sigma-model with Witten's Euclidean black hole metric coupled with non-simply laced ATT. Lie algebras associated with these "dual" Toda theories belong to the dual series of affine algebras but have a smaller rank equal to $r-1$.

To describe the action of the dual theory we introduce the complex scalar field $\chi = \chi_1 + i\chi_2$ and Toda field ϕ with $r-1$ components: $\phi = (\phi_1, ..., \phi_{r-1})$. In terms of these fields the dual action has the form:

$$\widetilde{\mathcal{A}}_r^{(\sigma)} = \int d^2x\left[\frac{1}{2\pi\widetilde{b}^2}\frac{\partial_\mu\chi\partial_\mu\overline{\chi}}{1+\chi\overline{\chi}} + \frac{(\partial_\mu\phi)^2}{8\pi} + 2\widetilde{\mu}\chi\overline{\chi}e^{\widetilde{b}e_{r-1}\cdot\phi} + \widetilde{\mu}\sum_{i=1}^{r-1}e^{\widetilde{b}e_i\cdot\phi} + \widetilde{U}_\sigma\right] \tag{122}$$

where e_i are the simple roots of Lie algebra B_{r-1}, $\widetilde{b} = 1/b$, and dual integrable perturbations are:

$$\widetilde{U}_1(\phi) = \widetilde{\mu}_1 e^{-\widetilde{b}\phi_1}; \quad \widetilde{U}_2(\phi) = \widetilde{\mu}_2 e^{-2\widetilde{b}\phi_1}; \quad \widetilde{U}_2(\phi) = \widetilde{\mu}_3 e^{-\widetilde{b}(\phi_1+\phi_2)}. \tag{123}$$

We see that the charged particles in these QFTs being weakly coupled fermions (at small b) flowing to the strong coupling region (small $\widetilde{b}$) take one degree of freedom from Toda lattice and transform to the weakly coupled bosons. This property of integrable QFTs (121,122) is used in condensed matter physics for nonperturbative analysis of superconductors coupled to phonons living in an insulating layer (see e.g. [40]). For nonperturbative analysis of these QFTs we need besides the FST data, which were described in ([39]), also the CFT data, characterizing the background CFTs.

The corresponding CFTs are described by the actions $\mathcal{A}_r^{(0)}$ eq.(118) and $\widetilde{\mathcal{A}}_r^{(0)}$ eq.(122) with $U_\sigma = \widetilde{U}_\sigma = 0$. From the duality of perturbed theories it follows that these CFTs are also dual i.e. describe the same theory. The CFT (122) for $r = 1$ was used in papers [41],[42] for the description of the string propagating in a black hole background. For arbitrary r these CFTs, known as non-abelian Toda theories, were considered in [43] as the models for extended black holes. In many cases, however, the dual representation with action $\mathcal{A}_r^{(0)}$, which can be called Sine-Toda theory is more convenient for the analysis.

The conformal invariance of Sine-Toda theories is generated by holomorphic stress-energy tensor:

$$T(z) = -\frac{1}{2}\left(\partial_z \Phi\right)^2 - \frac{1}{2}(\partial_z \varphi)^2 + Q\partial_z^2 \varphi \tag{124}$$

with $Q = \rho/b + b\rho'$, where ρ and ρ' are the Weyl vectors of Lie algebras B_r and D_r respectively. The exponential fields (for simplicity we consider the spinless fields):

$$V_{a,\eta}(x) = \exp(a \cdot \varphi + i\eta\Phi) \tag{125}$$

are conformal primary field with dimensions:

$$\Delta(a, \eta) = \frac{1}{2}(\eta^2 + a\,(Q - a)). \tag{126}$$

In particular, fields $V_{be_i,0}; i = 1, ..., r-1, V_{be_r,\pm\gamma}$ have conformal dimensions equal to one.

Besides the conformal symmetry, generated by $T \equiv T_2$ these CFTs possess an infinite-dimensional symmetry generated by the chiral algebra $\mathcal{T}$, which includes an infinite number of holomorphic fields T_j with integer spins. The detailed description of the chiral algebra $\mathcal{T}$ is not in the scope of this paper. As an example, we give here the spin-3 field $T_3 \in \mathcal{T}$ for the case $r = 1$. In this case the theory is described by two fields Φ and φ_1 and can be called as Sine-Liouville CFT with an action:

$$\mathcal{A}_1^{(0)} = \int d^2x \left[\frac{(\partial_\mu \Phi)^2}{8\pi} + \frac{(\partial_\mu \varphi_1)^2}{8\pi} + 2\mu' \cos(\gamma\Phi) e^{b\cdot\varphi_1} \right]. \tag{127}$$

The holomorphic field T_3 for this CFT has a form:

$$\begin{aligned} T_3 \;\; = \;\; & \frac{1+3b^2}{1+b^2}(\partial_z\Phi)^3 + \frac{3b^2}{2}(\partial_z\Phi)(\partial_z\varphi_1)^2 + \frac{3b^3}{2}(\partial_z\Phi)(\partial_z^2\varphi_1) \\ & -\frac{3}{2}b(1+b^2)(\partial_z^2\Phi)(\partial_z\varphi_1) + \frac{1}{4}(1+b^2)(\partial_z^3\Phi). \end{aligned}$$

The other fields $T_j, j > 3$ for this theory can be obtained by fusion of the field T_3.

The fields T_j in general case can be represented as the differential polynomials of the fields $\partial_z\Phi, \partial_z\varphi$ of weight j. It means that the exponential fields (125) are the primary fields of chiral algebra $\mathcal{T}$. The corresponding eigenvalues $t_j(a, \eta)$ of the operators $T_{j,0}$ (zero Fourier components of currents T_j) possess reflection symmetry: $t_j(a, \eta) = t_j(s(a), \eta)$ with respect to action of Weyl group $\mathcal{W}: \; s(a) = Q + \hat{\mathbf{s}}(a - Q)$ of Lie algebra B_r. The fields $V_{a,\eta}$ and $V_{s(a),\eta}$ are then related by reflection amplitudes:

$$V_{a,\eta} = R_s(a, \eta) V_{s(a),\eta}. \tag{128}$$

For calculation of reflection amplitudes in Sine-Toda theory it is convenient to use the screening charges:

$$\widehat{Q}_i = \int d^2x \exp(be_i \cdot \varphi), i = 1, ..., r-1; \quad \widehat{Q}_\pm = \int d^2x \exp(be_r \cdot \varphi \pm i\gamma\Phi)$$

which commute with all generators of chiral algebra. The calculation of the reflection amplitudes $R_s(a, \eta)$ follows the lines of sections 2 and 5. Here we give the result. To describe it is convenient to denote as S (L) the set of the short (long) positive roots of Lie algebra B_r: $(S : \alpha > 0, \alpha^2 = 1), (L : \alpha > 0, \alpha^2 = 2)$. Then, for arbitrary reflection, $s(a)$ the reflection amplitude can be represented in the usual form:

$$R_s(a, \eta) = \frac{A_{s(a),\eta}}{A_{a,\eta}}, \tag{129}$$

and function $A_{a,\eta}$ is:

$$
\begin{aligned}
A_{a,\eta} \quad &= \quad \left(\frac{\pi\mu'}{2b^2}\right)^{2\overline{a}\cdot\omega_r/b} \quad \left(\pi\mu\gamma(b^2)\right)^{\overline{a}\cdot\rho'/b}\prod_{\alpha\in L}\Gamma(1-\overline{a}_\alpha b)\Gamma(1-\overline{a}_\alpha/b) \\
&\times\prod_{\alpha\in S}\frac{\Gamma(1-2\overline{a}_\alpha b)\Gamma(1-\overline{a}_\alpha/b)}{\Gamma(1/2-\overline{a}_\alpha b+\gamma\eta)\Gamma(1/2-\overline{a}_\alpha b-\gamma\eta)}
\end{aligned} \tag{130}
$$

where ω_r is the fundamental weight of $B_r : 2\omega_r \cdot e_i = \delta_{r,i}$; ρ' is the Weyl vector of Lie algebra D_r and $\overline{a}_\alpha = (a-Q)\cdot\alpha$.

It was noted in the beginning of this section that in the semiclassical limit $b \to 0$ the Sine-Toda CFT is effectively described by C_r NATT and decoupled free field Φ. In agreement with this in the limit $b \to 0$ with $\overline{a}/b$ fixed the reflection amplitudes (129),(130) do not depend on parameter η and coincide with reflection amplitudes for C_r NATT calculated in [5].

Two points functions $D_r(a,\eta)$ of the fields $V_{a,\eta}$, normalized by the condition $D_r(a,\eta)D_r(2Q-a,\eta) = 1$ can be written as:

$$
D_r(a,\eta) = |x|^{4\Delta}\left\langle V_{a,\eta}(x), V_{a,-\eta}(0)\right\rangle = \frac{A_{2Q-a,\eta}}{A_{a,\eta}}. \tag{131}
$$

Until now we considered spinless fields and non-compactified field Φ. For string theory applications it is, however, important to use the periodicity property of Sine-Toda theories and to compactify the field Φ at the circle of length $2\pi/\gamma$, i.e. $\Phi = \Phi + 2\pi k/\gamma, k \in Z$. In this case the parameter η is quantized: $\eta_n = \gamma n$. It is useful to introduce the "dual" field Φ', defined by the relation: $\partial_\mu\Phi = \varepsilon_{\mu\nu}\partial_\nu\Phi'$ and to consider the local exponential fields with spin $\sigma = nm$, $m \in Z$, which can be written as:

$$
V(a,\eta_n,\eta'_m) = \exp(a\cdot\varphi)\exp(i\eta_n\Phi + i\eta'_m\Phi') \tag{132}
$$

where $\eta_n = \gamma n$ and $\eta'_k = m/2\gamma$. In particular, field χ in non-Abelian Toda theories (122) can be represented in terms of the fields of Sine-Toda theories as:

$$
\chi \sim \exp(-e_r\cdot\varphi/2b)\exp(i\Phi'/2\gamma). \tag{133}
$$

In the string theories associated with these CFTs the numbers n and m correspond to the winding number and momentum of sting propagating in the black hole background. The total momentum is conserved. The total winding number is not conserved and the sum of winding numbers of the operators in p-point functions can take the values between $2-p$ and $p-2$.

To obtain the reflection amplitudes and two point functions for the fields $V(a,\eta_n,\eta'_m)$ we should do the following substitution in the eq.(130). Namely, the dependence on parameter η there appears in two Γ-function which are in the denominator of the product over short roots. We should substitute $\eta \to \eta_n + \eta'_m$ in the argument of the first Γ-function and $\eta \to \eta_n - \eta'_m$ in the argument of the second one. For example, for the Sine-Liouville theory ($r = 1$ and $\overline{a} = \overline{a}_1$) we obtain:

$$\begin{aligned} D_1(a, \eta_n, \eta'_m) \quad = \quad & \left(\frac{\pi\mu'}{2b^2}\right)^{-2\overline{a}/b} \frac{\Gamma(1+2\overline{a}b)\Gamma(1+\overline{a}/b)}{\Gamma(1-2\overline{a}b)\Gamma(1-\overline{a}/b)} \times \\ & \frac{\Gamma(1/2-\overline{a}b+\gamma(\eta_n+\eta'_m))\Gamma(1/2-\overline{a}b-\gamma(\eta_n-\eta'_m))}{\Gamma(1/2+\overline{a}b+\gamma(\eta_n+\eta'_m))\Gamma(1/2+\overline{a}b-\gamma(\eta_n-\eta'_m))}. \end{aligned} \tag{134}$$

This two point function as well as duality between Sine-Liouville and Witten's 2D black hole models were obtained in collaboration with A.Zamolodchikov and Al.Zamolodchikov.

We note that besides the string theory, where two point functions of the vertex operators $V(a, \eta_n, \eta'_m)$ contain the information about the spectrum [42] and partition function of the theory [44], reflection amplitudes, derived in this section, can be used for the calculation of one point functions and UV asymptotics in massive QFTs (118,121,122). We suppose to discuss these problems in the separate publications.

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